Spherical Coordinates

In the examples considered so far, Cartesian coordinates were clearly appropriate, since the boundaries were *planes*. For *round* objects spherical coordinates are more natural. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{3.53}$$

I shall assume the problem has **azimuthal symmetry**, so that V *is independent of* ϕ;[7] in that case Eq. 3.53 reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \tag{3.54}$$

As before, we look for solutions that are products:

$$V(r,\theta)=R(r)\Theta(\theta). \tag{3.55}$$

Putting this into Eq. 3.54, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0. \tag{3.56}$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1). \tag{3.57}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a *partial* differential equation (3.54) into *ordinary* differential equations (3.57). The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R, \tag{3.58}$$

has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \tag{3.59}$$

as you can easily check; A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. But the angular equation,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta, \tag{3.60}$$

is not so simple. The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta). \tag{3.61}$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l. \tag{3.62}$$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	=	1
$P_1(x)$	=	x
$P_2(x)$	=	$(3x^2-1)/2$
$P_3(x)$	=	$(5x^3-3x)/2$
$P_4(x)$	=	$(35x^4-30x^2+3)/8$
$P_5(x)$	=	$(63x^5-70x^3+15x)/8$

Table 3.1 Legendre Polynomials

Notice that $P_l(x)$ is (as the name suggests) an lth-order *polynomial* in x; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front ($1/2^l l!$) was chosen in order that

$$P_l(1) = 1. \tag{3.63}$$

The Rodrigues formula obviously works only for nonnegative integer values of l. Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second*-order, and it should possess *two* independent solutions, for *every* value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.[8] For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right). \tag{3.64}$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general *separable* solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l. The *general* solution is the linear combination of separable solutions:

$$\boxed{V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).} \tag{3.65}$$

Example 3.6

The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Solution: In this case $B_l = 0$ for all l—otherwise the potential would blow up at the origin. Thus,

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{3.66}$$

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta). \tag{3.67}$$

Can this equation be satisfied, for an appropriate choice of coefficients A_l? *Yes:* The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \leq x \leq 1$

$(0 \leq \theta \leq \pi)$. How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are *orthogonal* functions:[9]

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases} \tag{3.68}$$

Thus, multiplying Eq. 3.67 by $P_{l'}(\cos\theta)\sin\theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta\, d\theta. \tag{3.69}$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69.

In can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 "by eyeball."[10] For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = k \sin^2(\theta/2), \tag{3.70}$$

where k is a constant. Using the half-angle formula, we rewrite this as

$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)].$$

Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other A_l's vanish. Evidently,

$$V(r, \theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R} P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right). \tag{3.71}$$

Example 3.7

The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.

Solution: In this case it's the A_l's that must be zero (or else V would not go to zero at ∞), so

$$V(r,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.72}$$

At the surface of the sphere we require that

$$V(R,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta).$$

Multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating—exploiting, again, the orthogonality relation 3.68—we have

$$\frac{B_{l'}}{R^{l'+1}} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta.$$

or

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.73}$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Example 3.8

An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. [The field will push positive charge to the "northern" surface of the sphere, leaving a negative charge on the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere.] Find the potential in the region outside the sphere.

Solution: The sphere is an equipotential—we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does *not* go to zero at large z. In fact, far from the sphere the field is $E_0\hat{\mathbf{z}}$, and hence

$$V \to -E_0 z + C.$$

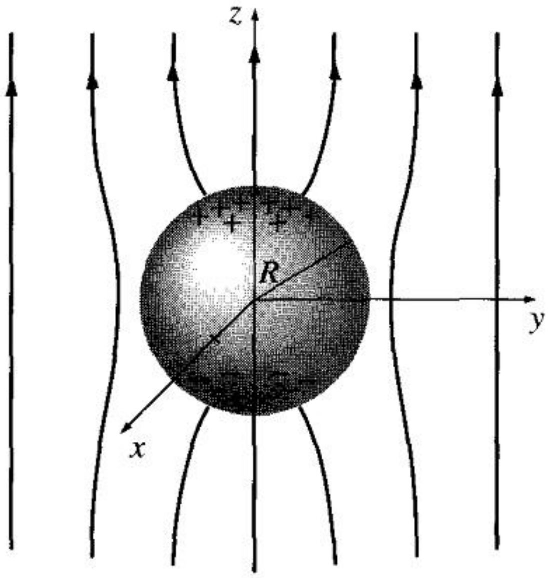

Figure 3.24

Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

$$\left.\begin{array}{ll}\text{(i)} & V = 0 \quad \text{when } r = R, \\ \text{(ii)} & V \to -E_0 r\cos\theta \quad \text{for } r \gg R.\end{array}\right\} \tag{3.74}$$

We must fit these boundary conditions with a function of the form 3.65.

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0,$$

or

$$B_l = -A_l R^{2l+1}, \tag{3.75}$$

so

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta).$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Evidently, only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0, \quad \text{all other } A_l\text{'s zero.}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.76}$$

The first term $(-E_0 r\cos\theta)$ is due to the external field: the contribution attributable to the induced charge is evidently

$$E_0\frac{R^3}{r^2}\cos\theta.$$

If you want to know the induced charge density, it can be calculated in the usual way:

$$\sigma(\theta) = -\epsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=R} = \epsilon_0 E_0\left.\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\right|_{r=R} = 3\epsilon_0 E_0\cos\theta. \tag{3.77}$$

As expected, it is positive in the "northern" hemisphere $(0 \le \theta \le \pi/2)$ and negative in the "southern" $(\pi/2 \le \theta \le \pi)$.

Example 3.9

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Solution: You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma_0}{\mathscr{r}}\,da,$$

but separation of variables is often easier. For the interior region we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R) \tag{3.78}$$

(no B_l terms—they blow up at the origin); in the exterior region

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \qquad (r \ge R) \tag{3.79}$$

(no A_l terms—they don't go to zero at infinity). These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is *continuous* at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta). \tag{3.80}$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta),$$

or, using Eq. 3.81:

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.83}$$

From here, the coefficients can be determined using Fourier's trick:

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.84}$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by Eqs. 3.81 and 3.84.

For instance, if

$$\sigma_0(\theta) = k\cos\theta = k P_1(\cos\theta), \tag{3.85}$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\epsilon_0}\int_0^{\pi}[P_1(\cos\theta)]^2 \sin\theta\, d\theta = \frac{k}{3\epsilon_0}.$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \le R), \tag{3.86}$$

whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \ge R). \tag{3.87}$$

$$E_0 \frac{R^3}{r^2}\cos\theta,$$

consistent with our conclusion in Ex. 3.8.

Problem 3.18 The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta,$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.)

SOLUTION :

Kita uraikan fungsi sinus menjadi

$$V_0(\theta) = k\cos(3\theta) = k\left[4\cos^3\theta - 3\cos\theta\right] = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right].$$

Kita peroleh

$$4\cos^3\theta - 3\cos\theta = \alpha\left[\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] + \beta\cos\theta = \frac{5\alpha}{2}\cos^3\theta + \left(\beta - \frac{3}{2}\alpha\right)\cos\theta,$$

Sehingga

$$4 = \frac{5\alpha}{2} \Rightarrow \alpha = \frac{8}{5}; \quad -3 = \beta - \frac{3}{2}\alpha = \beta - \frac{3}{2}\cdot\frac{8}{5} = \beta - \frac{12}{5} \Rightarrow \beta = \frac{12}{5} - 3 = -\frac{3}{5}.$$

Dengan demikian

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), & \text{for } r \le R \quad \text{(Eq. 3.66)} \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), & \text{for } r \ge R \quad \text{(Eq. 3.71)} \end{cases}$$

Dimana

$$\begin{aligned} A_l &= \frac{(2l+1)}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \quad \text{(Eq. 3.69)} \\ &= \frac{(2l+1)}{2R^l} \frac{k}{5} \left\{ 8 \int_0^{\pi} P_3(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta - 3 \int_0^{\pi} P_1(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta \right\} \\ &= \frac{k}{5} \frac{(2l+1)}{2R^l} \left\{ 8 \frac{2}{(2l+1)} \delta_{l3} - 3 \frac{2}{(2l+1)} \delta_{l1} \right\} = \frac{k}{5} \frac{1}{R^l} [8\,\delta_{l3} - 3\,\delta_{l1}] \\ &= \begin{cases} 8k/5R^3, & \text{if } l = 3 \\ -3k/5R, & \text{if } l = 1 \end{cases} \text{(zero otherwise).} \end{aligned}$$

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \qquad (3.69)$$

Dengan demikian

$$V(r,\theta) = -\frac{3k}{5R} r P_1(\cos\theta) + \frac{8k}{5R^3} r^3 P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right) P_1(\cos\theta)\right]},$$

Atau

$$\frac{k}{5}\left\{8\left(\frac{r}{R}\right)^3 \frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right] - 3\left(\frac{r}{R}\right)\cos\theta\right\} \Rightarrow \boxed{V(r,\theta) = \frac{k}{5}\frac{r}{R}\cos\theta\left\{4\left(\frac{r}{R}\right)^2\left[5\cos^2\theta - 3\right] - 3\right\}}$$

(untuk $r \le R$).

Sementara

$$B_l = A_l R^{2l+1}. \qquad (3.81)$$

Sehingga

$$B_l = \left\{ \begin{array}{ll} 8kR^4/5, & \text{if } l = 3 \\ -3kR^2/5, & \text{if } l = 1 \end{array} \right\}$$ Lainnya nol

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

Sehingga

$$V(r,\theta) = \frac{-3kR^2}{5}\frac{1}{r^2}P_1(\cos\theta) + \frac{8kR^4}{5}\frac{1}{r^4}P_3(\cos\theta) = \boxed{\frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]},$$

Atau

$$\boxed{V(r,\theta) = \frac{k}{5}\left(\frac{R}{r}\right)^2 \cos\theta \left\{4\left(\frac{R}{r}\right)^2 \left[5\cos^2\theta - 3\right] - 3\right\}}$$

(untuk r ≥ R). Akhirnya dengan menggunakan Persamaan 3.83

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

$$\left.\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta),$$

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.83}$$

diperoleh

$$\begin{aligned} \sigma(\theta) &= \epsilon_0 \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \epsilon_0\left[3A_1P_1 + 7A_3R^2P_3\right] \\ &= \epsilon_0\left[3\left(-\frac{3k}{5R}\right)P_1 + 7\left(\frac{8k}{5R^3}\right)R^2P_3\right] = \boxed{\frac{\epsilon_0 k}{5R}\left[-9P_1(\cos\theta) + 56P_3(\cos\theta)\right]} \\ &= \frac{\epsilon_0 k}{5R}\left[-9\cos\theta + \frac{56}{2}\left(5\cos^3\theta - 3\cos\theta\right)\right] = \frac{\epsilon_0 k}{5R}\cos\theta[-9 + 28\cdot 5\cos^2\theta - 28\cdot 3] \\ &= \boxed{\frac{\epsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right].} \end{aligned}$$

CATATAN :

$\cos 2\alpha = 1 - 2 \sin^2 \alpha$

$\cos 3\alpha = 4 \cos^3\alpha - 3 \cos\alpha$

$\cos 4\alpha = 8 \cos^4\alpha - 8 \cos^2\alpha + 1$

$\cos 5\alpha = 16 \cos^5\alpha - 20 \cos^3\alpha + 5 \cos\alpha$

$\cos (\alpha \pm \beta) = \cos\alpha \cos\beta -(\pm) \sin\alpha \sin\beta$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	=	1
$P_1(x)$	=	x
$P_2(x)$	=	$(3x^2 - 1)/2$
$P_3(x)$	=	$(5x^3 - 3x)/2$
$P_4(x)$	=	$(35x^4 - 30x^2 + 3)/8$
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Table 3.1 Legendre Polynomials